Group actions of prime order on local normal rings

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14 April 2011

In the theory of singularities, an important class of singularities is built by the famous Hirzebruch-Jung singularities. They arise by dividing out a finite cyclic group action on a smooth surface. The resolution of these singularities is well understood and have nice arithmetic properties related to continued fractions; cf. [H] and [J].
One can also look at such group actions from a purely algebraic point of view. So let B be a regular local ring and G a finite cyclic group of order n acting faithfully on B by local automorphisms. In the tame case; i.e. the order of G is prime to the characteristic of the residue field k of B , there is a central result of J.P. Serre [S1] saying that the action is given by multiplying a suitable system of parameters $(y_1, \ldots, y_d)$ by roots of unity $y_i \longmapsto \zeta^{n_i} \cdot y_i$ for $i = 1, \ldots, d$ where ζ is a primitive n^{th}-root of unity. Moreover, the ring of invariants $A := B^G$ is regular if and only if $n_i \equiv 0 \mod n$ for $d-1$ of the parameters. The latter is equivalent to the fact that $\mathrm{rk}((\sigma - \mathrm{id})|T) \leq 1$ for the action of $\sigma \in G$ on the tangent space $T := \mathfrak{m}_B/\mathfrak{m}_B^2$. For more details see [B, Chap. 5, ex. 7].
Only very little is known in the case of a wild group action; i.e., $\gcd(n, \mathrm{char}\, k) > 1$. In this paper we will restrict ourselves to the case of p-cyclic group actions; i.e. $n = p$ is a prime number. We will present a sufficient condition for the fact that the ring of invariants A is regular. Our result is also valid in the tame case; i.e. where n is a prime different from $\mathrm{char}\, k$. As the method of Serre depends on an intrinsic formula for writing down the action explicitly, we provide also an explicit formula for presenting B as a free A-module if our condition is fulfilled.
The interest in our problem stems from the investigation on the relationship between the regular and the stable R-model of a smooth projective curve X_K over the field of fractions K of a discrete valuation ring R . In general, the curve X_K admits a stable model X' over a finite Galois extension $R \to R'$. Then the Galois group $G = G(R'/R)$ acts on X' . Our result provides a means to construct a regular model over R starting from the stable model X' . We intend to work this out in a further article.

Finally, we want to mention that S. Wewers obtained partial results of ours by different methods cf. [W]

In this paper we will study only local actions of a cyclic group G of prime order p on a normal local ring B . We fix a generator σ of G and obtain the *augmentation map*

$$I := I_\sigma := \sigma - \mathrm{id} : B \longrightarrow B \ ; \ b \longmapsto \sigma(b) - b \ .$$

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We introduce the B-ideal

$$I_G := (I(b)\ ;\ b \in B) \subset B$$

which is generated by the image $I(B)$. This ideal is called *augmentation ideal.* If this ideal is generated by an element $I(y)$, we call y an *augmentation generator.* Note that this ideal does not depend on the chosen generator σ of G . Moreover, if y is an augmentation generator with respect to a generator σ of G , then y is also an augmentation generator for any other generator of G . Since B is local, the ideal I_G is generated by an augmentation generator if I_G is principal. Namely, $I_G/\mathfrak{m}_B I_G$ is a vector space over the residue field $k_B = B/\mathfrak{m}_B$ of B of dimension 1 . So it is generated by the residue class of $I(y)$ for some $y \in B$ and, hence, due to Nakayama's Lemma, I_G is generated by $I(y)$.

Definition 1 An action of a group G on a regular local ring B by local automorphisms is called a *pseudo-reflection* if there exists a system of parameters $(y_1, \ldots, y_d)$ of B such that $y_2, \ldots, y_d$ are invariant under G .

Theorem 2 *Let B be a normal local ring with residue field $k_B := B/\mathfrak{m}_B$. Let p be a prime number and G a p-cyclic group of local automorphisms of B . Let I_G be the augmentation ideal. Let A be the ring of G-invariants of B . Consider the following conditions:*

(a) $I_G := B \cdot I(B)$ *is principal.*

(b) B *is a monogenous* A*-algebra.*

(c) B *is a free* A*-module.*

Then the following implications are true:

$$(\mathrm{a}) \longleftrightarrow (\mathrm{b}) \longrightarrow (\mathrm{c})$$

Assume, in addition, that B is regular. Consider the following conditions:

(d) A *is regular.*

(e) G *acts as a pseudo-reflection.*

Then the condition (c) *implies* (d).

Moreover if, in addition, the canonical map $k_A \xrightarrow{\sim} k_B$ is an isomorphism. Then condition (a) *is equivalent to condition* (e).

We start the proof of the theorem by several preparations.

Remark 3 For $b_1, b_2, b \in B$, the following relations are true:

(i) $I(b_1 \cdot b_2) = I(b_1) \cdot \sigma(b_2) + b_1 \cdot I(b_2)$

(ii) $I(b^n) = \left(\sum_{i=1}^{n} \sigma(b)^{i-1} b^{n-i} \right) \cdot I(b)$

(iii) $I\left(\frac{b_1}{b_2}\right) = \frac{I(b_1)b_2 - b_1 I(b_2)}{b_2 \sigma(b_2)}$ if $b_2 \neq 0$.

Proof. (i) follows by a direct calculation and (ii) by induction from (i).
(iii) The formula (i) holds for elements in the field of fractions as well. Therefore it holds

$$I(b_1) = I\left(\frac{b_1}{b_2} b_2\right) = I\left(\frac{b_1}{b_2}\right)\sigma(b_2) + \frac{b_1}{b_2} I(b_2)$$

and the formula follows. □

For the implication (a) → (b) we need a technical lemma.

Lemma 4 *Let $y \in B$ be an augmentation generator. Then set, inductively,*

$$\begin{aligned} y_i^{(0)} &:= y^i && \text{for } i = 0, \ldots, p-1 \\ y_i^{(1)} &:= \frac{I\left(y_i^{(0)}\right)}{I\left(y_1^{(0)}\right)} && \text{for } i = 1, \ldots, p-1 \\ y_i^{(n+1)} &:= \frac{I\left(y_i^{(n)}\right)}{I\left(y_{n+1}^{(n)}\right)} && \text{for } i = n+1, \ldots, p-1 \ . \end{aligned}$$

Then

$$y_i^{(n)} = \sum_{0 \le k_1 \le \ldots \le k_{i-n} \le n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \quad \text{for } i = n, \ldots, p-1$$

and, in particular,

$$\begin{aligned} y_n^{(n)} &= 1 \\ y_{n+1}^{(n)} &= \sum_{j=1}^{n+1} \sigma^{j-1}(y) \\ I\left(y_{n+1}^{(n)}\right) &= \sigma^{n+1}(y) - y \end{aligned}$$

Furthermore, $y_{n+1}^{(n)}$ is again an augmentation generator for $n = 0, \ldots, p-2$.

Proof. We proceed by induction on n . For $n = 0$ the formulae are obviously correct. For the convenience of the reader we also display the formulae for $n = 1$. Due to Remark 3 one has

$$\begin{aligned} y_i^{(1)} &= \frac{I\left(y_i^{(0)}\right)}{I\left(y_1^{(0)}\right)} = \frac{I(y^i)}{I(y)} = \sum_{j=1}^{i} \sigma(y)^{j-1} y^{i-j} \\ &= \sum_{0 \le k_1 \le \ldots \le k_{i-1} \le 1} \prod_{\nu=1}^{i-1} \sigma^{k_\nu}(y) \end{aligned}$$

since the last sum can be viewed as a sum over an index j where $i-j$ is the number of the $k_\nu = 0$. In particular, the formulae are correct for $y_1^{(1)}$ and $y_2^{(1)}$. Moreover

$$I\left(y_2^{(1)}\right) = I(\sigma(y) - y) = \sigma^2(y) - y \ .$$

Since σ^2 is generator of G for $2 < p$, the element $y_2^{(1)}$ is an augmentation generator as well. Now assume that the formulae are correct for n . Since $y_{n+1}^{(n)}$ is an augmentation generator, $I\left(y_{n+1}^{(n)}\right)$ divides $I\left(y_i^{(n)}\right)$ for $i = n+1, \ldots, p-1$. Then it remains to show

$$I\left(y_i^{(n)}\right) = \left(\sigma^{n+1}(y) - y\right) \cdot y_i^{(n+1)} \quad \text{for} \quad i = n+1, \ldots, p-1 \quad .$$

For the left hand side one computes

$$\begin{aligned} LHS &= I\left(\sum_{0\le k_1\le\ldots\le k_{i-n}\le n} \prod_{j=1}^{i-n} \sigma^{k_j}(y)\right) = \sum_{0\le k_1\le\ldots\le k_{i-n}\le n} I\left(\prod_{j=1}^{i-n} \sigma^{k_j}(y)\right) \\ &= \sum_{0\le k_1\le\ldots\le k_{i-n}\le n} \cdot \left(\prod_{j=1}^{i-n} \sigma^{k_j+1}(y) - \prod_{j=1}^{i-n} \sigma^{k_j}(y)\right) \\ &= \sum_{1\le k_1\le\ldots\le k_{i-n}\le n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \; - \sum_{0\le k_1\le\ldots\le k_{i-n}\le n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \quad . \end{aligned}$$

Now all terms occurring in both sums cancel. These are the terms with $k_{i-n} \le n$ in the first sum and $1 \le k_1$ in the second sum.
For the right hand side one computes

$$\begin{aligned} RHS &= \left(\sigma^{n+1}(y) - y\right) \cdot \sum_{0\le k_1\le\ldots\le k_{i-n-1}\le n+1} \prod_{j=1}^{i-n-1} \sigma^{k_j}(y) \\ &= \sum_{0\le k_1\le\ldots\le k_{i-n}=n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \; - \sum_{0=k_1\le\ldots\le k_{i-n}\le n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \quad . \end{aligned}$$

Comparing both sides one obtains $LHS = RHS$. In particular we have

$$\begin{aligned} y_{n+1}^{(n+1)} &= 1 \\ y_{n+2}^{(n+1)} &= \sum_{0\le k_1\le n+1} \prod_{j=1}^{1} \sigma^{k_1}(y) = \sum_{j=1}^{n+2} \sigma^{j-1}(y) \\ I\left(y_{n+2}^{(n+1)}\right) &= \sigma^{n+2}(y) - y \quad . \end{aligned}$$

So $y_{n+2}^{(n+1)}$ is an augmentation generator for $n+2 < p$, since σ^{n+2} generates G . This concludes the technical part. $\square$

Proposition 5 *Assume that the augmentation ideal* I_G *is principal and let* $y \in B$ *be an augmentation generator. Then* B *decomposes into the direct sum*

$$B = A \cdot y^0 \oplus A \cdot y^1 \oplus \ldots \oplus A \cdot y^{p-1} \quad .$$

Proof. Since $I(y) \neq 0$, the element y generates the field of fractions $Q(B)$ over $Q(A)$. Therefore

$$Q(B) = Q(A) \cdot y^0 \oplus Q(A) \cdot y^1 \oplus \ldots \oplus Q(A) \cdot y^{p-1} \quad .$$

Then it suffices to show the following claim:

Let $a, a_0, \ldots, a_{p-1} \in A$. Assume that a divides

$$b = a_0 \cdot y^0 + a_1 \cdot y^1 + \ldots + a_{p-1} \cdot y^{p-1} \quad .$$

Then a divides $a_0, a_1, \ldots, a_{p-1}$.

If $b = a \cdot \beta$, then $I(b) = a \cdot I(\beta)$. Since $I(\beta) = \beta_1 \cdot I(y)$, we get $I(b) = a\beta_1 \cdot I(y)$. So we see that a divides $I(b)/I(y) \in B$. Using the notations of Lemma 4, set

$$\begin{array}{rcccl} b^{(0)} & := & b & = & a_0 \cdot y^0 + a_1 \cdot y^1 + \ldots a_{p-1} \cdot y^{p-1} \\ b^{(1)} & := & \dfrac{I\left(b^{(0)}\right)}{I(y)} & = & a_1 + a_2 \dfrac{I\left(y^2\right)}{I(y)} + \ldots + a_{p-1} \dfrac{I\left(y^{p-1}\right)}{I(y)} \\ & & & = & a_1 \cdot y_1^{(1)} + a_2 \cdot y_2^{(1)} + \ldots + a_{p-1} \cdot y_{p-1}^{(1)} \\ b^{(n)} & := & \dfrac{I\left(b^{(n-1)}\right)}{I\left(y_n^{(n-1)}\right)} & = & a_n \cdot y_n^{(n)} + a_{n+1} \cdot y_{n+1}^{(n)} + \ldots + a_{p-1} \cdot y_{p-1}^{(n)} \quad . \end{array}$$

Due to the observation above, we see by induction that a divides $b^{(0)}, b^{(1)}, \ldots, b^{(p-1)}$, since $y_{n+1}^{(n)}$ is an augmentation generator for $n = 1, \ldots, p-2$. So we obtain

$$a \,\big|\, b^{(p-1)} = a_{p-1} \cdot y_{p-1}^{(p-1)} = a_{p-1} \quad .$$

Now proceeding downwards, one obtains

$$a \,\big|\, b^{(p-2)} = a_{p-2} + a_{p-1} \cdot y_{p-1}^{(p-2)} \text{ and, hence, } a \,\big|\, a_{p-2}$$
$$a \,\big|\, b^{(n)} = a_n + a_{n+1} \cdot y_{n+1}^{(n)} + \ldots + a_{p-1} \cdot y_{p-1}^{(n)} \text{ and, hence, } a \,\big|\, a_n$$

for $n = p-1, p-2, \ldots, 0$. □

Proof of the first part of Theorem 2.

(a) → (b): This follows from Proposition 5.
(b) → (a): If $B = A[y]$ is monogenous, then $I_G = B \cdot I(y)$ is principal.
(b) → (c) is clear. Namely, if $B = A[y]$, the minimal polynomial of y over the field of fraction is of degree p and the coefficients of this polynomial belong to A . Then B has $y^0, y^1, , \ldots, y^{p-1}$ as an A-basis.

Next we do some preparations for proving the second part of the theorem where B is assumed to be regular.

Lemma 6 *Let R be a local subring of B which is invariant under G such that the canonical map $R/\mathfrak{m}_R \xrightarrow{\sim} B/\mathfrak{m}_B$ is an isomorphism. Let $(y_1, \ldots, y_d)$ be a generating system of the maximal ideal $\mathfrak{m}_B$. Then I_G is generated by $I(y_1), \ldots, I(y_d)$.*

Proof. Due to the assumption, we have $B = R + \mathfrak{m}_B$ and, hence, $I(B) = I(\mathfrak{m}_B)$. Furthermore, we have

$$\mathfrak{m}_B = \mathfrak{m}_B^2 + \sum_{i=1}^{d} R \cdot y_i \quad .$$

Since I is R-linear, we get

$$I(\mathfrak{m}_B) = I\left(\mathfrak{m}_B^2\right) + \sum_{i=1}^{d} R \cdot I(y_i) \ .$$

Due to Remark 3, one knows $I\left(\mathfrak{m}_B^2\right) \subset \mathfrak{m}_B \cdot I(\mathfrak{m}_B)$. So one obtains

$$I(\mathfrak{m}_B) \subset \mathfrak{m}_B \cdot I(\mathfrak{m}_B) + \sum_{i=1}^{d} R \cdot I(y_i) \ .$$

Since B is local, Nakayama's Lemma yields

$$I_G = B \cdot I(B) = B \cdot I(\mathfrak{m}_B) = \sum_{i=1}^{d} B \cdot I(y_i) \ .$$

Thus the assertion is proved. $\square$

Proposition 7 *Keep the assumption of the second part of the theorem; namely that B is regular and that the canonical morphism $k_A \xrightarrow{\sim} k_B$ is an isomorphism. Let $(y_1, \ldots, y_d)$ be a generating system of the maximal ideal $\mathfrak{m}_B$. Then the following assertions are true:*

(i) $I_G = B \cdot I(y_1) + \ldots + B \cdot I(y_d)$

(ii) *If the ideal $I_G = B \cdot I(B)$ is principal, then there exists an index $i \in \{1, \ldots, d\}$ with $I_G = B \cdot I(y_i)$.*

Proof. Let $\widehat{B}$ be the $\mathfrak{m}_B$-adic completion of B . Recall that a regular ring is noetherian by definition. Therefore the extension $B \to \widehat{B}$ is faithfully flat and $\mathfrak{m}_B\widehat{B} = \mathfrak{m}_{\widehat{B}}$; cf. [AM, 10.14 & 10.15]. Since G acts by local morphism, any $\sigma \in G$ extends to a local automorphism $\widehat{\sigma}$ of $\widehat{B}$. Due to the assumption that the canonical morphism $k_A \xrightarrow{\sim} k_B$ is an isomorphism, any $b \in B$ can be written as $B = a + m$ where $a \in A$ is invariant under G and $m \in \mathfrak{m}_B$ and, hence, $I(b) = I(m) \in I(\mathfrak{m}_B)$. Therefore I_G is generated by $I(\mathfrak{m}_B)$.
(i) Since $B \to \widehat{B}$ is faithfully flat and $\widehat{B} \cdot \mathfrak{m}_B = \mathfrak{m}_{\widehat{B}}$, it suffices to prove the assertion for the completion $\widehat{B}$. For complete local rings there exists a G-stable lift R of the residue field k . Namely, in the case of mixed characteristic $(0, p)$, one can choose the ring of Witt vectors $W(k) \subset \widehat{A}$ as R and, in the equal characteristic case, the residue field k lifts into $\widehat{A}$; cf. [C]. Now we can apply Lemma 6 and obtain the assertion.
(ii) Since I_G is principal, $I_G/\mathfrak{m}_B I_G$ is generated by one of the $I(y_i)$ and, hence, again by Nakayama's Lemma $I_G = B \cdot I(y_i)$ for a suitable $i \in \{1, \ldots, d\}$. $\square$

Proof of the second part of Theorem 2.

(c) $\to$ (d) follows from [M, Theorem 51]. Namely, B is noetherian due to the definition of a regular ring. Since $A \to B$ is faithfully flat, so A is noetherian. Then one can apply loc.cit.
(a) $\to$ (e) We assume that the canonical map $k_A \to k_B$ of the residue fields is an isomorphism. If I_G is principal, one can choose an augmentation generator $y \in \mathfrak{m}_B$ which is part of a system of parameters $(y, y_2, \ldots, y_d)$ due to Proposition 7. Due to Proposition 5, we know that B decomposes into the direct sum

$$B = A \cdot y^0 \oplus A \cdot y^1 \oplus \ldots \oplus A \cdot y^{p-1} \ .$$

Now we can represent

$$y_j = \sum_{i=0}^{p-1} a_{i,j} \cdot y^i \text{ for } j = 2, \ldots, d \ .$$

Then set

$$\tilde{y}_j := y_j - \sum_{i=1}^{p-1} a_{i,j} y^i = a_{0,j} \in A \cap \mathfrak{m}_B = \mathfrak{m}_A \text{ for } j = 2, \ldots, d \ .$$

So $(y, \tilde{y}_2, \ldots, \tilde{y}_d)$ is a system of parameters of B as well. Thus G acts by a pseudo-reflection.
(e) $\to$ (a): If G is a pseudo-reflection, I_G is generated by $I(y)$ due to Proposition 7 where $y, x_2, \ldots, x_p$ is a system of parameters with $x_i \in \mathfrak{m}_A$ for $i = 2, \ldots, p$ if $k_A = k_B$. □

If $k_A \to k_B$ is not an isomorphism, the implication (e) $\to$ (a) is false as the following shows.

Example 8 Let k be a field of positive characteristic p and look at the polynomial ring

$$R := k[Z, Y, X_1, X_2]$$

over k . We define a p -cyclic action of $G = \langle \sigma \rangle$ on R by

$$\sigma | k := \mathrm{id}_k \, , \, \sigma(Z) = Z + X_1 \, , \, \sigma(Y) = Y + X_2 \, , \, \sigma(X_i) = X_i \text{ for } i = 1, 2 \ .$$

This is a well-defined action of order p , since $p \cdot X_i = 0$ for $i = 1, 2$, and it leaves the ideal $\mathfrak{I} := (Y, X_1, X_2)$ invariant. Furthermore, for any $g \in k[Z] - \{0\}$ the image is given by $\sigma(g) = g + I(g)$ with $I(g) \in X_1 \cdot k[Z, X_1]$.
Then consider the polynomial ring

$$S := k(Z)[Y, X_1, X_2]$$

over the field of fractions $k(Z)$ of the polynomial ring $k[Z]$. Then S has the maximal ideal $\mathfrak{m} = (Y, X_1, X_2)$. Then set

$$B := S_{\mathfrak{m}} = k(Z)[Y, X_1, X_2]_{(Y, X_1, X_2)} \ .$$

We can regard all these rings as subrings of the field of fractions of R

$$R \subset S \subset B \subset k(Z, Y, X_1, X_2) \ .$$

Clearly, σ acts on R and, hence, it induces an action on its field of fractions; denote this action by σ as well. Then we claim that the restriction of σ to B induces an action on B by local automorphisms. For this, it suffices to show that for any $g \in R - \mathfrak{I}$ the image $\sigma(g)$ does not belong to $\mathfrak{I}$. The latter is true, since

$$\sigma(g) = g + I(g) \text{ with } I(g) \in \mathfrak{I} \ .$$

The augmentation ideal $I_G = B \cdot X_1 + B \cdot X_2$ is not principal although G acts through a pseudo-reflection. □

Remark 9 In the tame case $p \neq \mathrm{char}(k_B)$, the converse (d) $\to$ (a) is also true due to the theorem of Serre as explained in the introduction.

In the case of a wild group action; i.e. $p = \mathrm{char}(k_B)$, it is not known whether the converse is true, but we would conjecture it.

Conjecture 10 Let B be a regular local ring and let G be a p-cyclic group acting on B by local automorphisms. Then the following conditions are *conjectured* to be equivalent:

(1) I_G is principal.

(2) $A := B^G$ is regular.

The implication $(1) \to (2)$ was shown in Theorem 2. Of course the converse is true if $\dim A \leq 1$. In higher dimension, the converse $(2) \to (1)$ is uncertain, but it holds for small primes $p \leq 3$ as we explain now. Since A is regular, the ring B is a free A-module of rank p ; cf. [S2, IV, Prop. 22]. So,

$(*)$ $\qquad B/B\mathfrak{m}_A^n$ is a free $A/\mathfrak{m}_A^n$-module of rank p for any $n \in \mathbb{N}$.

In the case $p = 2$ the rank of $\mathfrak{m}_B/B\mathfrak{m}_A$ is 0 or 1 . In the first case, k_B is an extension of degree $[k_B : k_A] = 2$ over k_A and $\mathfrak{m}_B = B\mathfrak{m}_A$. So there exists an element $\beta \in B$ such that $B/B\mathfrak{m}_A$ is generated by the residue classes of 1 and β . Due to Nakayama's Lemma $B = A[\beta]$ is monogenous and, hence, I_G is principal. In the second case, where $k_A \to k_B$ is an isomorphism, then there exists an element $\beta \in \mathfrak{m}_B$ such that $\mathfrak{m}_B = B\beta + B\mathfrak{m}_A$. Then G acts as a pseudo-reflection and, hence, I_G is principal.

In the case $p = 3$ we claim that $B\mathfrak{m}_A \not\subset \mathfrak{m}_B^2$!
If we assume the contrary $B\mathfrak{m}_A \subset \mathfrak{m}_B^2$ then these ideals coincide; $B\mathfrak{m}_A = \mathfrak{m}_B^2$. Namely, the rank of $B/B\mathfrak{m}_A$ as $A/\mathfrak{m}_A$-module is 3 and the rank of $B/\mathfrak{m}_B^2$ is at least 3 due to $d := \dim B \geq 2$, so $B\mathfrak{m}_A = \mathfrak{m}_B^2$. Therefore the length of $B/B\mathfrak{m}_A^2 = B/\mathfrak{m}_B^4$ is 3 times the length of $A/\mathfrak{m}_A^2$ which is $3 \cdot (\dim A + 1)$. On the other hand the rank of $B/\mathfrak{m}_B^4$ is equal to

$$\left(1 + \dim \mathfrak{m}_B/\mathfrak{m}_B^2\right) + \dim \mathfrak{m}_B^2/\mathfrak{m}_B^3 + \dim \mathfrak{m}_B^3/\mathfrak{m}_B^4 = \sum_{n=0}^{3} \binom{d+n-1}{d-1}$$

which larger than

$$\left(1 + \dim \mathfrak{m}_A/\mathfrak{m}_A^2\right) + \left(1 + \dim \mathfrak{m}_A/\mathfrak{m}_A^2\right) + \left(1 + \dim \mathfrak{m}_A/\mathfrak{m}_A^2\right)$$

since for $d \geq 2$ holds

$$\binom{d+1}{d-1} = \frac{(d+1)d}{2} \geq 1 + d = 1 + \dim \mathfrak{m}_A/\mathfrak{m}_A^2$$

and

$$\binom{d+3-1}{d-1} = \frac{(d+2)(d+1)d}{2 \cdot 3} > 1 + d$$

Here we used the formula for the number $\lambda_{n,d}$ of monomials $T_1^{m_1} \ldots T_d^{m_d}$ in d variables of degree $n = m_1 + \ldots + m_d$

$$\lambda_{n,d} = \binom{d+n-1}{d-1} .$$

So, using only the condition $(*)$ and proceeding by induction on $\dim(A)$, we see that here exists a system of parameters $\alpha_1, \ldots, \alpha_d$ of A such that $\alpha_2, \ldots, \alpha_d$ is part of a system of parameters of B. In the case, where $k_A \to k_B$ is an isomorphism, G acts as a pseudo-reflection and, hence, I_G is principal. If $k_A \to k_B$ is not an isomorphism, then we must have $\mathfrak{m}_B = B\mathfrak{m}_A$; otherwise the rank of $B/\mathfrak{m}_B$ is at least 4. Since $[k_B : k_A] \leq 3$, the field extension $k_A \to k_B$ is monogenous and, hence, $A \to B$ is monogenous due to the Lemma of Nakayama. □

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